The Influence of Maintenance on Degradation Processes and the Resulting Lifetime Distribution

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Degradation processes

Lifetime of products or biological organisms often are affected by degradation (damage, stress, shocks).

$\rightarrow$ Consider the development of a stochastic process which models ageing, wearing, damage accumulation, and other changes of the state (degradation models).

Purely lifetime based reliability analysis is limited

- Collecting a sample of lifetime data is often expensive.
- In applications from engineering, observed lifetimes are frequently very long.
- There are many products for which break-down causes serious difficulties.

When designing the mathematical model of a degradation process, the following steps have to be taken:

1. Starting from basic technical considerations a suitable mathematical model for the degradation measure process $Z(t)$ has to be defined.
2. By applying the level exceeding theory it is possible to find the type of lifetime distribution, i.e. the distribution of the time at which the stochastic process firstly exceeds the degradation level.
3. The parameters of the selected model have to be estimated by means of samples related to the degradation measure $Z(t)$.
4. Estimations of the parameters of lifetime distribution can be obtained from the estimated parameters of the model $Z(t)$, if the limit level of degradation is known.

Degradation processes

Most common degradation models:

- Deterministic pathes:
  Linear path model (Meeker)

  $$Z(t) = t/A \quad \text{where} \quad A \quad \text{is a positive random variable}$$

  Generalization: Non-parametric methods of estimation from joint linear degradation and failure time data (Bagdonavcius, Bikelis, Kazakevicius, Nikulin)
- Gamma degradation process: a continuous-time stochastic process with independent and gamma distributed increments (Bagdonavcius, Nikulin), (van Noortwijk).

Degradation processes

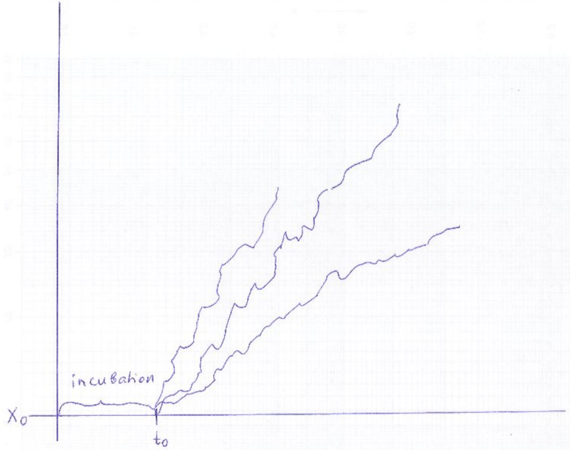
- Diffusion processes (K., Lehmann, Doksum and Normand) (Whitmore: include measurement errors), (Whitmore and Schenkelberg: time scale transformations), (Whitmore, Crowder and Lawless: bivariate models), (Gaschler: Ornstein-Uhlenbeck process)
- Shock models: Marked point processes (Sobczyk, Esary,Marshall,Proshan, K.,Wendt, Anderson, Alsmeyer, Gut,Hüsler, . . .)

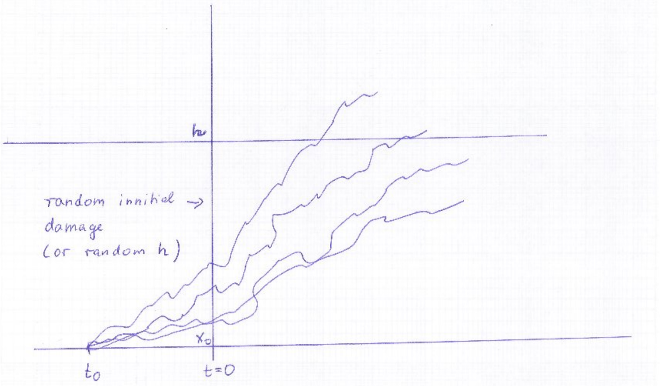
Degradation Process: Wiener process with drift $Z(t)$:

$$Z(t) = z_0 + \sigma W(t - t_0) + \mu \cdot (t - t_0) \,, \quad t \geq t_0 \tag{1}$$

with

z_0	-	constant initial degradation ($z_0 \in \mathbb{R}$),
t_0	-	beginning of the degradation ($t_0 \in \mathbb{R}$),
μ	-	drift parameter ($\mu \in \mathbb{R}$),
σ	-	variance parameter ($\sigma > 0$),
$W(t)$	-	standard Wiener process on $[0, \infty)$.


incubation
X_0
t_0


h
random initial → damage (or random h)
x_0
t_0
$t=0$

The lifetime T_h

$$T_h = \inf\{t \geq t_0 : Z(t) \geq h\} \ . \tag{2}$$

Inverse Gaussian distribution:

$$f_{T_h}(t) = \frac{h - z_0}{\sqrt{2\pi\sigma^2(t - t_0)^3}} \exp\left(-\frac{(h - z_0 - \mu(t - t_0))^2}{2\sigma^2(t - t_0)}\right) I_{\{t>t_0\}}, \tag{3}$$

Degradation Process: Gamma Process $X(t)$:

$(X_t)_{t\geq 0}$ -stochastic process in continuous time with independent and gamma distributed increments $X_t - X_s \sim \Gamma(a(t-s), b)$

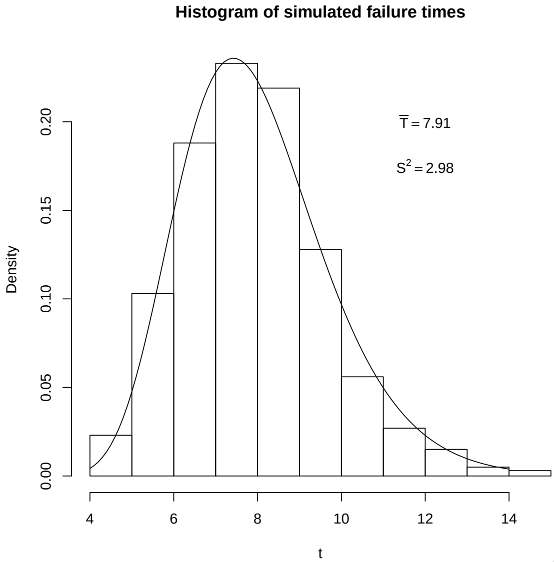
$$E(X_t) = \frac{a}{b}t, \qquad Var(X_t) = \frac{a}{b^2}t$$

Advantage: monoton increasing, it is easy to simulate.

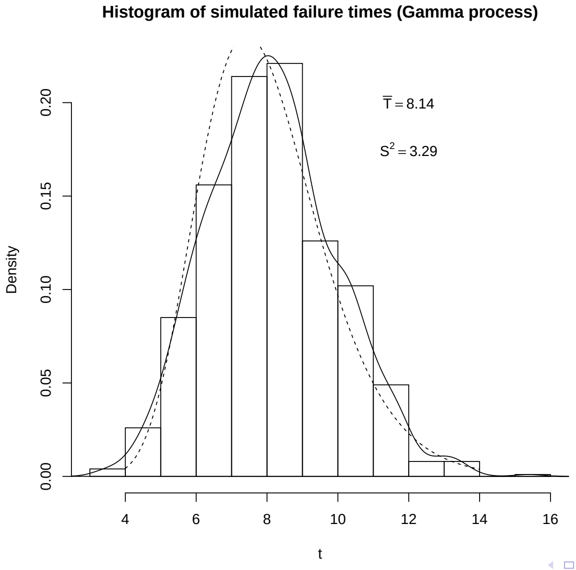
Disadvantage: Parameter estimation is more complicated, the first passage time is given implicitely:

$$P(T_h > t) = P(X_h < h) = \int_0^h \frac{1}{\Gamma(at)} b^{at} x^{at-1} e^{-bx} dx$$

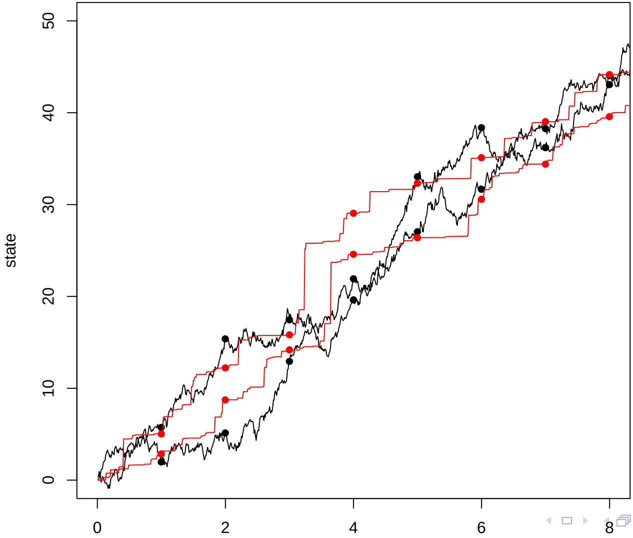
Simulated failure times (Wiener processs $\mu = 5, \sigma^2 = 10, h = 40$)



Simulated failure times (Gamma process $a = 2.5$, $b = 0.5$, $h = 40$)



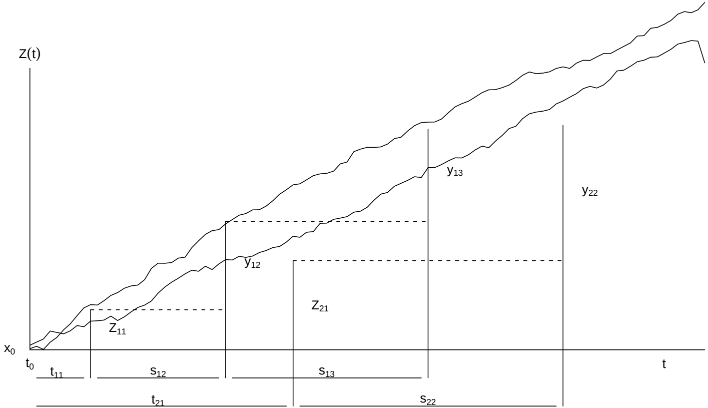
Process realizations



Parameter estimation

Parameters μ, σ^2, t_0, and z_0 may be estimated before a failure occurs.

- n realizations of the degradation process $Z_i(t)$, $i = 1, \ldots, n$ are observed.
- From each realization m_i observations $Z_i(t_{ij})$ at times t_{ij}, $j = 1, \ldots, m_i$; $i = 1, \ldots, n$ are given.


Z(t)
y13
y22
y12
Z21
Z11
x0
t0
t11
s12
s13
t
t21
s22

The likelihood function

$$L = \prod_{i=1}^{n} \frac{1}{\sqrt{\sigma^2(t_{i1}-t_0)}} \varphi\left(\frac{z_{i1}-z_0-\mu(t_{i1}-t_0)}{\sqrt{\sigma^2(t_{i1}-t_0)}}\right) \prod_{j=2}^{m_i} \frac{1}{\sqrt{\sigma^2 s_{ij}}} \varphi\left(\frac{(y_{ij}-\mu s_{ij})}{\sqrt{\sigma^2 s_{ij}}}\right), \quad (4)$$

where φ is the density function of the standard normal distribution.

In this case , which is formally obtained on setting $h = \infty$, we make neither use of information on failure times nor of the fact that the degradation process does not exceed the level h between two observation points. Note that the number of observations and the observation points may be different in each realization.

Essential Simplification

The first observation points are equal for all realizations:
$t_{i1} = t_1, \; i = 1, \ldots, n.$
explicit solution of the likelihood equations:

$$\begin{aligned}
\hat{\mu} &= (\bar{z}_{\cdot m} - \bar{z}_{\cdot 1}) / (\bar{t}_{\cdot m} - t_1) \,, \\
\hat{\sigma}^2 &= \frac{1}{\bar{m}(n-1)} \left(\sum_{i=1}^{n} \sum_{j=2}^{m_i} \frac{y_{ij}^2}{s_{ij}} - n \frac{(\bar{z}_{\cdot m} - \bar{z}_{\cdot 1})^2}{\bar{t}_{\cdot m} - t_1} \right) \,, \\
\hat{z}_0 &= \left(\bar{z}_{\cdot 1} (t_{\cdot m} - \hat{t}_0) - \bar{z}_{\cdot m} (t_1 - \hat{t}_0) \right) / (t_{\cdot m} - t_1) \,, \\
\hat{t}_0 &= t_1 - \frac{1}{\hat{\sigma}^2} \left(\sum_{i=1}^{n} z_{i1}^2 - \bar{z}_{\cdot 1}^2 \right) \,. \qquad (5)
\end{aligned}$$

(μ, σ^2) are parameters of interest

New nuisance parameters ν_1, ν_2 can be found, which are independent from the parametes of interest (μ, σ^2):

$$\begin{aligned} \nu_1 &= z_0 + \mu(t_1 - t_0) \,, \\ \nu_2 &= \sigma^2(t_1 - t_0) \,. \end{aligned}$$

These new parameters decribe the expectation and the variance of the degradation process at the first observation time t_1.

likelihood function with new nuisance parameters

$$L = \left(\prod_{i=1}^{n} \frac{1}{\sqrt{\nu_2}} \varphi\left(\frac{z_{i1} - \nu_1)}{\sqrt{\nu_2}} \right) \right) \left(\prod_{i=1}^{n} \prod_{j=2}^{m_i} \frac{1}{\sqrt{\sigma^2 s_{ij}}} \varphi\left(\frac{(y_{ij} - \mu s_{ij}}{\sqrt{\sigma^2 s_{ij}}} \right) \right) , \tag{6}$$

2 independent parts

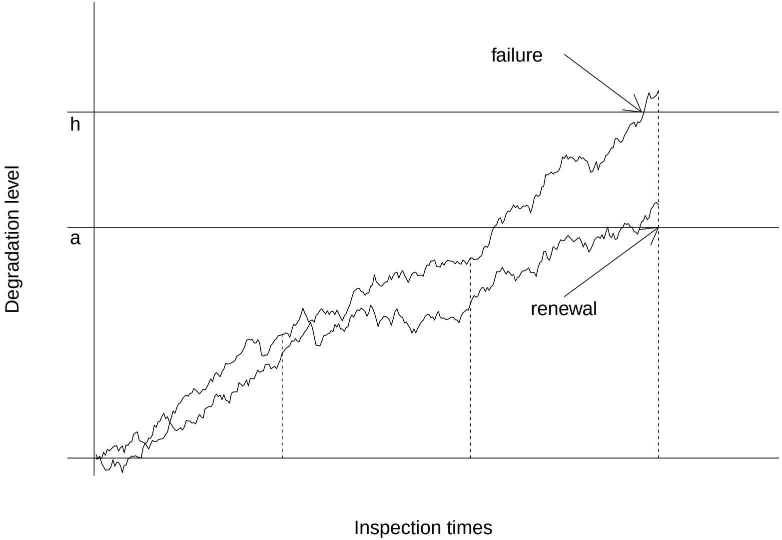
Structure of observation (K, Lehmann)

- $t_1, \ldots, t_m$: fixed observation points with

$$t_0 < t_1 < \ldots < t_m < \infty$$

.

- A failure is observable at any time $t > t_0$
- After a failure has occured we stop observing the degradation process
- Hence, in each time interval $(t_{j-1}, t_j]$ we observe either a failure or we observe the degradation measure t_j under the condition that the process has not yet exceeded the level h until the time t_j


failure
h
a
renewal
Degradation level
Inspection times

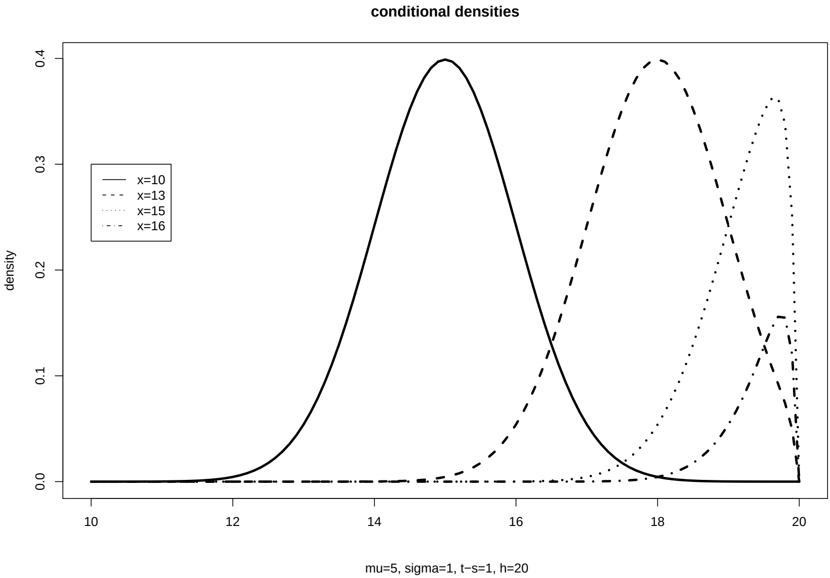
The density of the conditional distribution

Lemma

Let $t_0 \leq s < t$, $x < h$, and $z \in \mathbb{R}$. The conditional density is given by

$$f(s,x,t,z,h) = \frac{1}{\sigma\sqrt{t-s}}\,\phi\left(\frac{z-x-\mu(t-s)}{\sigma\sqrt{t-s}}\right) \times \left[1-\exp\left(-\frac{2(h-x)(h-z)}{\sigma^2(t-s)}\right)\right] I_{\{z\leq h\}}, \quad (7)$$

where ϕ is the density of the standard normal distribution.


conditional densities
density
0.0
0.1
0.2
0.3
0.4
x=10
x=13
x=15
x=16
10
12
14
16
18
20
mu=5, sigma=1, t−s=1, h=20

For preventive maintenance:
inspections of the degradation at times δ, 2δ, $\dots$.

If the degradation is larger than a predefined level a: preventive renewal.

Three kinds of costs:

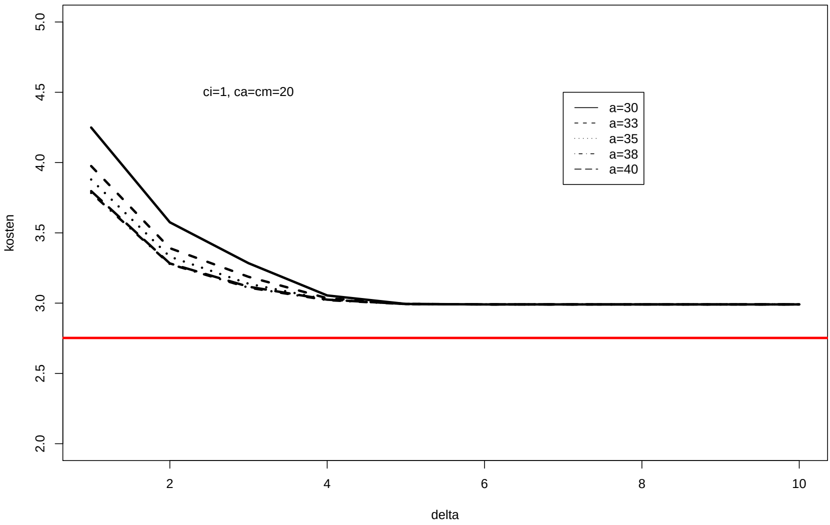
- costs of inspection c_I,
- costs of (preventive) maintenance c_M,
- costs of a failure c_A.

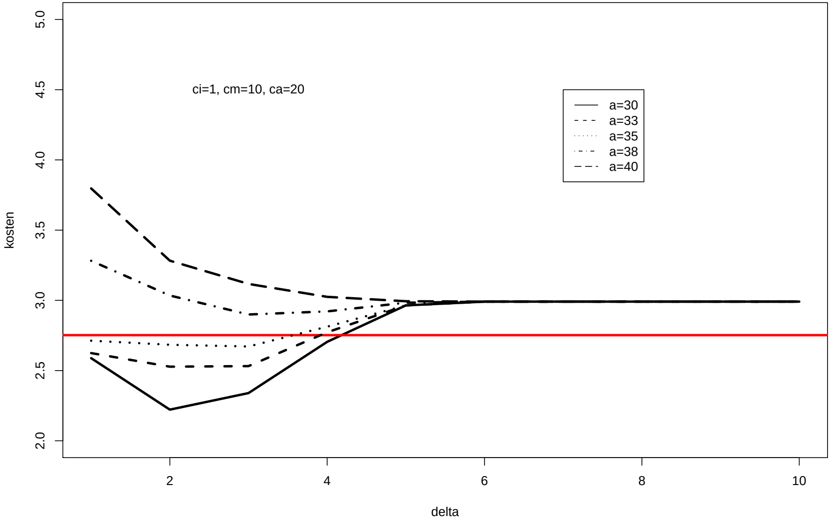
Average costs per time unit:

$$K = \frac{E(j * c_I + c_M I_M + c_A I_A)}{\text{expected length of a cycle}}$$

where

- j: number of inspections,
- $I_M = 1$ if the cycle ends by a maintenance,
- $I_A = 1$ if the cycle ends by a failure.


ci=1, ca=cm=20
a=30
a=33
a=35
a=38
a=40
kosten
delta
5.0
4.5
4.0
3.5
3.0
2.5
2.0
2
4
6
8
10


ci=1, cm=10, ca=20
a=30
a=33
a=35
a=38
a=40
kosten
delta
2.0
2.5
3.0
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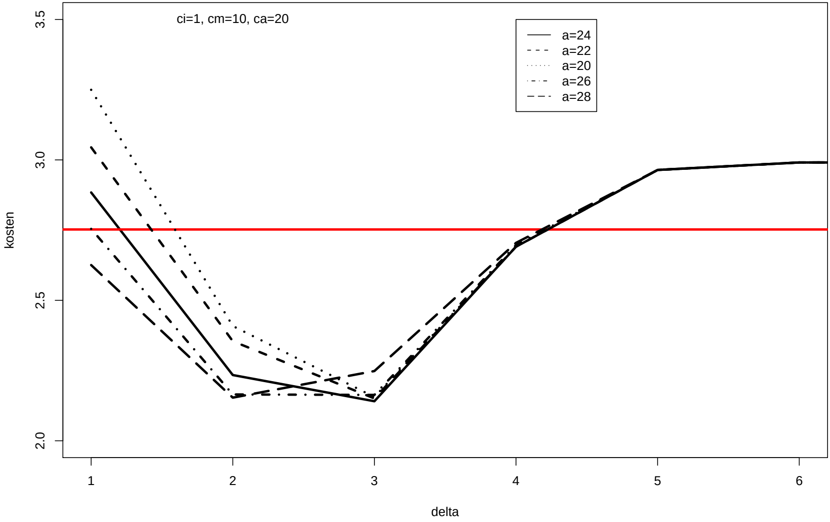
2
4
6
8
10


ci=1, cm=10, ca=20
a=24
a=22
a=20
a=26
a=28
kosten
delta
3.5
3.0
2.5
2.0
1
2
3
4
5
6

Kijima imperfect repair model: (after a failure)

- initial item with failure rate $\lambda_1(t) = \lambda(t)$.
- t_1: end of the first sojourn time (after a failure or after a shut down)
- the item will be repaired with the degree ξ_1
  minimal repair: $\xi_1 = 1$,
  perfect repair (renewal): $\xi_1 = 0$
  imperfect repair: $0 < \xi_1 (< 1)$
- The age of the item is decreased to

$$v_1 = \xi_1 t_1$$

  (virtual age of the item at time t_1)
- the distribution of the time until the next failure has the failure rate $\lambda_2(t) := \lambda(t - t_1 + v_1)$
- ....

Model specification

The process defined by

$$v(t) := t - t_n + v_n, \quad t_n \leq t < t_{n+1}\ ,\ n \geq 1$$

is called the virtual age process.

$$\begin{aligned} \text{Kijima Type I:} \quad v_k = & \quad v_{k-1} + \xi_k(t_k - t_{k-1}) \\ \text{Kijima Type II:} \quad v_k = & \quad \xi_k(v_{k-1} + (t_k - t_{k-1})) \end{aligned}$$

The distribution of the time until the next failure then has failure intensity $\lambda_{k+1}(t) = \lambda(t - t_k + v_k)$.

Application to Degradation Processes

We consider the degradation process $Z(t)$ with maintenance points $\tau_1, \tau_2, \ldots$
First inspection: The state of the process is $Z(\tau_1-) = z_1$.
Let ξ be the degree of repair.
There are 2 possibilities of an incomplete repair:

(a) reduction of degradation level : $z(\tau_1) = \xi \cdot z_1$
$\rightarrow$ the new virtual age is $v(\tau_1) = \xi \cdot z_1/\mu$

(b) reduction of virtual age $v(\tau_1) = \xi \cdot \tau_1$
$\rightarrow$ the new state is $z(\tau_1) = \xi \cdot \mu \cdot \tau_1$

Second inspection point: $Z(\tau_2-) = z_2$

(a) reduction of degradation level : $z(\tau_2) = \xi \cdot z_1 + \xi(z_2 - z_1)$
new virtual age: $v(\tau_2) = \xi \cdot z_2/\mu$ (Kijima Typ I) or

$z(\tau_2) = \xi \cdot (\xi \cdot z_1 + (z_2 - z_1)) = \xi^2 \cdot z_1 + \xi(z_2 - z_1)$
new virtual age: $v(\tau_2) = \xi^2 z_1/\mu + \xi(z_2 - z_1)/\mu$ (Kijima Typ II)

(b) reduction of virtual age $v(\tau_2) = \xi \cdot \tau_2 = \xi \cdot \tau_1 + \xi \cdot (\tau_1 - \tau_2)$
new state: $z(\tau_2) = \xi \cdot \mu \cdot \tau_2)$ (Kijima Typ I) or

$v(\tau_2) = \xi^2 \cdot \tau_1 + \xi \cdot (\tau_2 - \tau_1)$
new state: $z(\tau_2) = \xi^2 \cdot \mu \cdot \tau_1 + \xi\mu \cdot (\tau_1 - \tau_2)$ (Kijima Typ II)

Transformation (case (a)) for $\xi = .6$